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Rigidly rotating charged dust distribution in general relativity

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Abstract. Following two recent papers by Bonnor and Raychaudhuri on the motion of charged dust in general relativity, the authors obtain exact solutions of the Einstein–Maxwell equations in both cylindrically symmetric and axially symmetric cases.

1. Introduction

The present paper is a consequence of two recent papers by Bonnor (1980) and Raychaudhuri (1982). Raychaudhuri investigated the motion of charged dust in general relativity under the specific assumptions of rigid motion and that the electromagnetic potential vector (A^{μ}) and the velocity vector (V^{μ}) of the dust are everywhere coincident in direction, i.e. $A^{\mu} = kV^{\mu}$, where k is a scalar. Without introducing any symmetry assumptions he reduced the Einstein–Maxwell equations to relations which seem comparatively easy to solve. The present paper is an attempt to solve one such set of equations corresponding to a vanishing Poynting vector, first in the case of cylindrical symmetry and then in the more involved case of axial symmetry. The solutions are obtained by utilising the fact that the equation system is underdetermined and hence one can append an *ad hoc* relation.

2. The equation of the problem

The equations to be solved are given below in the notation of Raychaudhuri (1982):

$$\rho/\sigma = (1-kg)/g \tag{2.1}$$

$$4\pi\sigma = (1-kg)k_{;\mu}^{;\mu} - k(g'+g^2-f^2/k^2)k_{,\mu}k^{,\mu} \tag{2.2}$$

$$4\pi\rho[1-g^2/(1-kg)^2]$$
$$= \{(1-kg)^2 - g^2 + f^2 + (1-kg)^{-1}[g'+g^2-(f^2/2k^2)-f^2g/2k]\}k_{,\mu}k^{,\mu} \tag{2.3}$$

$$fk_{;\mu}^{;\mu} = -(f'+2fg-f/k)k_{,\mu}k^{,\mu} \tag{2.4}$$

$$E_{\mu} = (1-kg)k_{,\mu} \tag{2.5}$$

$$B_{\mu} = f(k)k_{,\mu} = -2k\omega_{\mu} \tag{2.6}$$

$$V_{\mu;\alpha}V^{\alpha} = \dot{V}_{\mu} = gk_{,\mu} = \lambda_{,\mu}/\lambda \tag{2.7}$$

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where ρ and σ are the matter density and charge density respectively, ω^{μ} is the vorticity vector, E_{μ} and B_{μ} are the electric and magnetic field vectors as seen by matter, $f(k)$ and $g(k)$ being functions of k, and the prime denotes differentiation with respect to k. In equation (2.3) a minor correction in the corresponding equation in Raychauduri (1982) has been made.

3. Solution of the equations in the case of cylindrical symmetry

We take the line element in the cylindrical symmetric form

$$ds^2 = F\,dt^2 - e^{2\psi}(dr^2 + dz^2) + 2m\,d\phi\,dt - l\,d\phi^2 \tag{3.1}$$

F, l, m and ψ being functions of r alone. We number the coordinates t, r, z, ϕ as 0, 1, 2, 3 respectively. The coordinate system is assumed co-moving, i.e.

$$V^{\mu} = \lambda\delta^{\mu}_{0} \tag{3.2}$$

where λ is the same as in equation (2.7). The normalisation condition on V^{μ} gives

$$F = 1/\lambda^2. \tag{3.3}$$

Now, corresponding to (3.1), $\dot{V}^2 = \dot{V}^3 = \dot{V}^0 = 0$, which implies, because of (2.7), that k (and hence λ) is also a function of r alone. We then obtain $\omega^1 = \omega^3 = \omega^0 = 0$ which implies from (2.6)

$$f = 0, \qquad \text{i.e. } B_{\mu} = 0. \tag{3.4}$$

But then $\sqrt{-g}\omega^2 = \frac{1}{2}V_0^2(m/F)_{,1} = 0$ and so

$$F = Am \qquad \text{where } A \text{ is a constant of integration.} \tag{3.5}$$

Again in view of (2.5) the only non-vanishing component of E_{μ} is E_1. Now with vanishing B_{μ} and S_{μ} (Poynting vector), the energy momentum tensor (Lichnerowicz 1967) $T^{\mu}_{\nu} = \rho V^{\mu}V_{\nu} - (1/4\pi)[(\frac{1}{2}\delta^{\mu}_{\nu} - V^{\mu}V_{\nu})(E^2 + B^2) + (E^{\mu}E_{\nu} + B^{\mu}B_{\nu}) + (V^{\mu}S_{\mu} + V_{\nu}S^{\mu})]$ gives $T^1_1 + T^2_2 = 0$ and hence because of Einstein's field equations $R^{\mu}_{\nu} = 8\pi(T^{\mu}_{\nu} - \frac{1}{2}T\delta^{\mu}_{\nu})$, $R^3_3 + R^0_0 = 0$. We can then introduce Weyl canonical coordinates such that

$$Fl + m^2 = r^2. \tag{3.6}$$

With $f = 0$, equation (2.4) is identically satisfied. Now from equations (2.1)–(2.3)

$$k_{;\mu}^{;\mu} = \{g(1-kg) + g'[g + k(1-kg)]/[(1-kg)^2 - g^2]\}k_{,\mu}k^{,\mu}. \tag{3.7}$$

Keeping in mind that only the derivatives with respect to r do not vanish, equation (3.7) gives on integration

$$rk_{,1} = B[g^2 - (1-kg)^2]^{-1/2} \tag{3.8}$$

where B is an arbitrary constant. Also from R^2_2 equation (van Stockum 1937)

$$(r\psi_{,1})_{,1} = \{[r(g^2 + g')(1-kg)]/[(1-kg)^2 - g^2]\}k^2_{,1}. \tag{3.9}$$

To obtain explicit integrals in terms of simple functions we utilise the freedom to introduce a relation between g and k, namely

$$gk = c\ (= \text{constant}). \tag{3.10}$$

Then equation (3.8) gives

$$r = D\left(\frac{(1-c)k\,\exp\{(1/c)[c^2-(1-c)^2k^2]^{1/2}\}}{c+[c^2-(1-c)^2k^2]^{1/2}}\right)^{c/B} \tag{3.11}$$

where D is an arbitrary constant of integration. Again from (3.8)–(3.10)

$$\psi = \ln(\alpha k^c r^\delta) \tag{3.12}$$

where α and δ are constants of integration.

The expression for matter density is

$$4\pi\rho = \frac{B^2(1-c)^2}{\alpha^2}\left(\frac{(1-c)^2k^2-c(c+1)}{[c^2-(1-c)^2k^2]^2}\right)\frac{k^{2(1-c)}}{r^{2(1+\delta)}} \tag{3.13}$$

and

$$\rho/\sigma = [(1-c)/c]k. \tag{3.14}$$

The metric components are given by (3.3), (3.5), (3.6) and (3.12) where

$$\lambda = \beta k^c \tag{3.15}$$

β being an arbitrary constant of integration.

4. Discussion of the nature of the solution in § 3

From (3.11), as $k \to 0$, $r \to k^{c/B}$. We choose c/B to be positive. Then $r \to 0$ implies $k \to 0$. Also since $k^2 \leqslant c^2/(1-c)^2$, we have to cut off the solution at a finite value of r, say $r_{\max} = D$. Of course, we can choose arbitrarily large values of D and hence the region of validity of the solution can be arbitrarily extended. Again from equation (3.12), $e^{2\psi} = \alpha^2 k^{2c} r^{2\delta}$. Thus as $r \to 0$ (i.e. $k \to 0$) $e^{2\psi} \to r^{2(B+\delta)}$. If $e^{2\psi}$ is to remain finite at $r = 0$, we must choose $B + \delta = 0$. Also, from equation (3.13), $4\pi\rho \to -[(1-c)^2(c+1)/c^3](B^2/\alpha^2)r^{2(B-c)}$ as $r \to 0$. Thus c must be negative such that $c \geqslant -1$. Also for ρ to remain finite at $r = 0$ we must have $B = c$.

From equations (3.5) and (3.15) and the fact that B has been chosen to be negative, we must find m and $F \to 0$ as $r \to 0$. From equation (3.6) if l is to vanish as $r \to 0$, we must choose $B > -1$. Thus, excepting the axis of symmetry, the solution obtained is regular up to arbitrarily large values of r.

It may be of interest to note that the result that the magnetic field vanishes follows even in the more general case of $V^\mu = \lambda(\delta^\mu_0 + \delta^\mu_3 + \delta^\mu_2)$. This particular fact is a consequence of the assumptions $A^\mu = kV^\mu$, $S^\mu = 0$ and the cylindrical symmetry of the line element.

The solution obtained differs from that of Som and Raychaudhuri (1968), in that in their case the Lorentz force vanishes and hence the motion is geodesic, whereas in our case the motion is not geodesic, i.e. acceleration exists though expansion and shear vanish.

Finally, since the vorticity vector vanishes, the velocity vector must be hypersurface orthogonal and the metric should be reducible to the static form, as is indeed possible by the transformation $dt' = dt + (1/A)\,d\phi$, in view of equation (3.5).

5. Solution of the equations in the case of axial symmetry

The line element is again taken as in (3.1), but now F, l, m and ψ are considered as functions of both r and z. Equation (3.2) and, consequently, equation (3.3) is assumed to hold. A direct calculation now gives

$$\dot{V}_1 = -\tfrac{1}{2}F_{,1}/F \qquad \dot{V}_2 = -\tfrac{1}{2}F_{,2}/F \qquad \dot{V}_3 = \dot{V}_0 = 0. \tag{5.1}$$

Thus, in view of equation (2.7), k and λ are also functions of r and z only. The only surviving components of ω^{μ} are

$$\omega^1 = -\tfrac{1}{2}(F/\sqrt{-g})(m/F)_{,2} \qquad \text{and} \qquad \omega^2 = \tfrac{1}{2}(F/\sqrt{-g})(m/F)_{,1}. \tag{5.2}$$

Thus equation (2.6) gives

$$(Fk/\sqrt{-g})(m/F)_{,2} = fk^{\cdot 1} \qquad (Fk/\sqrt{-g})(m/F)_{,1} = -fk^{\cdot 2}. \tag{5.3}$$

Equations (5.3) imply

$$k_{,2}(m/F)_{,2} + k_{,1}(m/F)_{,1} = 0. \tag{5.4}$$

Also, equations (5.1) and (2.7) yield

$$gk_{,1} = -\tfrac{1}{2}F_{,1}/F \qquad \text{and} \qquad gk_{,2} = -\tfrac{1}{2}F_{,2}/F. \tag{5.5}$$

Now, we assume

$$gk = \tfrac{1}{2} \tag{5.6}$$

(note that (5.6) is a special case of (3.10)). Equations (5.5) and (5.6) give

$$F = B^2/k, \tag{5.7}$$

B^2 being the integration constant. Also from equations (2.1), (2.2), (2.3) and (2.4) we get

$$f = \sqrt{k^2 - 1} \tag{5.8}$$

and

$$k_{;\mu}^{;\mu} = -[k/(k^2-1)]k_{,\mu}k^{,\mu}. \tag{5.9}$$

Equation (5.9) can be converted into the following Laplace equation

$$\chi_{,11} + \chi_{,22} + \chi_{,1}/r = 0 \tag{5.10}$$

where

$$\chi = \tfrac{1}{2}(k\sqrt{k^2-1} - \ln|k + \sqrt{k^2-1}|) \tag{5.11}$$

so that χ is real only if $k^2 \geqslant 1$.

Different axially symmetric solutions may be obtained by a different choice of solution of equation (5.10). We now try to solve (5.10) by the method of separation of variables. Thus let

$$\chi = R(r)Z(z),$$

so that (5.10) may be written in the form

$$R_{,11}/R + R_{,1}/rR + z_{,22}/z = 0.$$

Obviously we must have

$$z_{,22}/z = \alpha^2 \ (= \text{constant}) \tag{5.12}$$

and

$$R_{,11}/R + R_{,1}/rR = -\alpha^2. \tag{5.13}$$

For $\alpha \neq 0$ equation (5.12) and (5.13) have the integrals

$$Z = B \sinh(\alpha z) \qquad \text{and} \qquad R = AJ_0(\alpha r)$$

where J_0 is Bessel's function of order zero and A and B are constants of integration. The complete solution of χ is then given by

$$\chi = \beta J_0(\alpha r) \sinh(\alpha z) \tag{5.14}$$

where $\beta = AB$ = constant.

Now from equation (5.3), the expression for m follows

$$mk = -r\beta\alpha^{-1} J_0'(\alpha r) \cosh(\alpha z) \tag{5.15}$$

where the prime indicates differentiation with respect to r, and l can be found from the relation $Fl + m^2 = r^2$ which still holds. The determination of ψ is somewhat laborious. It involves appeal to the R_2^1 equation. We simply quote the result:

$$e^{2\psi} = k \exp[-2\beta^2 J_0(\alpha r) J_0'(\alpha r) \sinh^2(\alpha z)]. \tag{5.16}$$

The expression for density comes out as

$$4\pi\rho = -[(4k^4 - 9k^2 + 3)/4(k^2 - 1)^2]\, e^{-2\psi}(\chi_{,1}^2 + \chi_{,2}^2). \tag{5.17}$$

6. Discussion of the nature of the solution in § 5

First of all we may note from (5.17) that, since density is to be non-negative, we must restrict the solution to the space region for which

$$4k^4 - 9k^2 + 3 \leqslant 0.$$

This will happen when k^2 satisfies $\varepsilon_- \leqslant k^2 \leqslant \varepsilon_+$ where ε_- and ε_+ are the roots of $4x^2 - 9x + 3 = 0$. The values of ε_- and ε_+ are approximately 0.4 and 1.8, respectively. But since the condition of reality demands $k^2 \geqslant 1$, the range of k^2 is actually $1 \leqslant k^2 \leqslant \varepsilon_+$. At $k^2 = 1$, ρ becomes infinite, whereas at $k^2 = \varepsilon_+$, ρ becomes zero. When $k^2 = 1$, equations (5.11) and (5.14) show that for a finite z, r is the root of $J_0(\alpha r) = 0$. When $k^2 = \varepsilon_+$, the value of χ increases from zero and so for the same z, r is less than its value for $J_0(\alpha r) = 0$ as is evident from a consideration of the nature of $J_0(\alpha r)$.

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